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UNIQUENESS AND EXISTENCE OF WHITTAKER MODELS FOR THE METAPLICTIC GROUP

BY

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ABSTRACT

We introduce the notion of Whittaker models for representations of a metaplectic covering group of $GL(2)$ and establish the uniqueness and existence of such models. Our results generalize corresponding results of Jacquet–Langlands, but the methods are new.

1. Introduction

Our purpose is to give a new proof of the uniqueness and existence of Whittaker models for GL_2 and extend this result to the metaplectic group.

Let F denote a nonarchimedean local field and G the group $GL_2(F)$. The metaplectic group $\bar{G}$ is a non-trivial topological central extension of G by the group $\mathscr{Z}_2 = \{\pm 1\}$. If π is any representation of G, then π lifts naturally to a representation of $\bar{G}$ which is trivial on $\mathscr{Z}_2$. On the other hand, representations of $\bar{G}$ which are non-trivial on $\mathscr{Z}_2$ define projective (or multiplier) representations of G. Such representations of $\bar{G}$ are called *genuine*.

When convenient, we realize $\bar{G}$ as the set of pairs (g, ζ) with $g \in G$ and $\zeta \in \mathscr{Z}_2$. Multiplication is given by the formula

$$(g, \zeta)(g', \zeta') = (gg', \beta(g, g')\zeta\zeta')$$

with β an appropriate non-trivial two-cocycle on G; cf. [1] and [7]. If H is any subgroup of G, let $\bar{H}$ denote the complete inverse of H in $\bar{G}$. In particular, $\bar{Z}$ will denote the inverse image of the center Z of G. Note that $\bar{Z}$ is abelian, but not the center of $\bar{G}$.

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If $\bar{G}$ splits over H, there is a subgroup of $\bar{G}$ isomorphic to H which we again denote by H. In this case, $\bar{H} = H \times \mathscr{Z}_2$, a direct product. In particular, if

$$N = \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \right\}$$

then $\bar{N} = N \times \mathscr{Z}_2$. If

$$Z^2 = \left\{ \begin{bmatrix} \alpha^2 & 0 \\ 0 & \alpha^2 \end{bmatrix} : \alpha \in F^\times \right\}$$

then $\bar{G}$ also splits over Z^2, and $\bar{Z}^2$ is the center of $\bar{G}$.

Now fix a non-trivial character ψ of F. If (π, V) is an irreducible admissible representation of $\bar{G}$, let $\mathscr{L}_{(\pi,\psi)}$ denote the space of ψ-*Whittaker functionals* for (π, V), i.e., functionals ℓ on V such that

$$\ell\left(\pi\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} v\right) = \psi(x)\ell(v)$$

for all $x \in F$ and $v \in V$. According to a well-known result of [6], the dimension of $\mathscr{L}_{(\pi,\psi)}$ is at most one whenever π comes from an ordinary representation of GL_2. On the other hand, for arbitrary π this is no longer true, and we must proceed as follows.

Let ω_π denote the character of $(F^\times)^2$ defined by the equation

$$\pi\begin{pmatrix} z^2 & 0 \\ 0 & z^2 \end{pmatrix} = \omega_\pi(z^2)I, \qquad z \in F^\times.$$

Let $\Omega(\omega_\pi)$ denote the set of characters of $\bar{Z}$ whose restriction to $Z^2 \approx (F^\times)^2$ agrees with ω_π. This latter set is finite since the index of squares in $F^\times$ is finite. In any case, for each μ in $\Omega(\omega_\pi)$, define a character ψ^* on

$$N^* = \bar{Z} \times N$$

by setting $\psi^*(zn) = \mu(z)\psi(n)$. Then our purpose in this paper is to prove the following:

UNIQUENESS THEOREM. *For each μ in $\Omega(\omega_\pi)$, the space of linear functionals ℓ on V_π such that*

$$\ell(\pi(n^*)v) = \psi^*(n^*)\ell(v), \qquad n^* \in N^*, \quad v \in V_\pi,$$

is at most one dimensional.

We call such functionals (ψ, μ)-*Whittaker functionals* for (π, V).

EXISTENCE THEOREM. *If π is infinite dimensional, there is at least one μ in $\Omega(\omega_\pi)$ such that (π, V) admits a non-zero (ψ, μ) Whittaker functional.*

REMARKS. (i) Given π and ψ, let $\Omega(\pi, \psi)$ denote the set of μ in $\Omega(\omega_\pi)$ such that π admits a non-zero (ψ, μ)-functional. If $\Omega(\pi, \psi)$ is a singleton set we say π is *distinguished*. (Although the set $\Omega(\pi, \psi)$ depends on ψ, its cardinality does not.)

(ii) Every representation of GL_2 is distinguished in the above sense. Indeed suppose π on $\bar{G}$ comes from an ordinary representation of G. Then π operates as a scalar on all of $\bar{Z}$, and $\Omega(\pi, \psi)$ contains only one character — the "full central character" of π. In this case the requirement that ℓ be a Whittaker functional reduces to the equation

$$\ell(\pi(n)v) = \psi(n)\ell(v),$$

and our results above simply extend familiar results of Jacquet and Langlands.

(iii) If (π, V) is genuine, the assumption that π be infinite-dimensional is redundant; cf. Remark 5.2.

To prove our uniqueness theorem we establish the commutativity of a certain spherical function algebra on $\bar{G}$. This approach seems to be new even for GL_2; it is due to the second-named author, who sketched a proof (for GL_2) in 1973.

Both our theorems were first announced in [2] (where the term "exceptional" was used in place of "distinguished"). Consequences for the representation theory of $\bar{G}$ are discussed in [3] and applications to the theory of automorphic forms are described in [4] as well as [2]. The first named author wishes to thank the Math Departments of the Hebrew University and Tel Aviv University for their hospitality during the semester this work was completed.

2. Preliminaries

Let $(\cdot, \cdot)$ denote Hilbert's symbol in F. An explicit description of the co-cycle β defining multiplication in $\bar{G}$ can be found in [1] or [7]. For our purposes, it suffices to recall that

(1) Hilbert's symbol is trivial on $(F^\times)^2 \times (F^\times)^2$;

(2) on the group of upper triangular matrices,

$$\beta\left(\begin{bmatrix} a & x \\ 0 & b \end{bmatrix}, \quad \begin{bmatrix} a' & x' \\ 0 & b' \end{bmatrix}\right) = (a, b');$$

the parentheses on the right denote Hilbert's symbol.

In F, let O_F denote the ring of integers, $\mathcal{P}$ the prime ideal, and τ a fixed generator of $\mathcal{P}$. If a unit of O_F is congruent to 1 modulo $\mathcal{P}^m$, with m sufficiently large, then it is a square.

If $g \in G$, and we are working in the context of $\bar{G}$, we often write g for $(g, 1)$ in $\bar{G}$. The projection pr: $\bar{G} \to G$ is defined by the equation $\mathrm{pr}((g, \zeta)) = g$.

3. Commutativity of a certain spherical function algebra

Fix an additive character ψ of F and suppose $\psi(x) = 1$ if and only if $x \in O_F$.

For each $m \geqq 1$, let U_m denote the group of units congruent to 1 modulo $\mathcal{P}^m$. Given a character μ of $\bar{Z}$, fix m such that

$$\text{(i)} \qquad U_m \subset (F^\times)^2,$$

and

$$\text{(ii)} \qquad \mu\left(\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix}, 1\right) \equiv 1 \qquad \text{for all } z \in U_m.$$

According to the properties of the cocycle β, $\bar{G}$ splits over the group

$$P_m = \left\{\left(\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}, \zeta\right) : a, d \in U_m, b \in O_F\right\}.$$

Thus we may write

$$\bar{P}_m = P_m \times \mathcal{Z}_2,$$

and define a character ψ_m of $\bar{P}_m$ by the formula

$$\psi_m\left(\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}, \zeta\right) = \psi(\tau^{-m} b).$$

When convenient, we also regard ψ_m as a character of F.

Now consider the group

$$P_m^* = ZP_m.$$

Since everything in $\bar{Z}$ commutes with $\bar{P}_m$, the formula

$$\mu_m(zp_m) = \mu(z)\psi_m(p_m)$$

defines a character of $\overline{ZP_m} = \overline{P_m^*}$. (indeed μ and ψ_m are both trivial on $\bar{Z} \cap \overline{P_m}$.) Note that μ_m is "genuine" iff μ is genuine.

THEOREM 3.1. *Let $\mathscr{H}(\psi,\mu,m)$ denote the algebra (under convolution) of all locally constant functions on $\bar{G}$, compactly supported modulo Z^2, such that*

$$\varphi(pgp')=\mu_m(pp')\varphi(g)$$

for all $g\in\bar{G}$, and $p,p'\in\overline{P_m^*}$. Then $\mathscr{H}(\psi,\mu,m)$ is commutative.

To prove Theorem 3.1 we introduce an anti-automorphism of $\bar{G}$ of order two which fixes each $\varphi\in\mathscr{H}(\psi,\mu,m)$. Then familiar arguments (due to Gelfand) imply $\varphi_1*\varphi_2=\varphi_2*\varphi_1$; cf. [8], §II.1.

For simplicity, *we assume μ is genuine*, and treat the GL_2 case only parenthetically. (Cf. the Remark at the end of this Section.)

LEMMA 3.2. *There exists a map $\sigma\colon\bar{G}\to\bar{G}$ such that*

(i) *σ is an anti-automorphism;*

(ii) $\sigma^2=I$;

(iii)
$$\mathrm{pr}\left(\left(\begin{bmatrix}a&b\\c&d\end{bmatrix},\zeta\right)^{\sigma}\right)=\begin{pmatrix}d&b\\c&a\end{pmatrix},$$

i.e., σ "lifts" the anti-automorphism of G given by

$$g\to\begin{pmatrix}0&1\\1&0\end{pmatrix}g^{t}\begin{pmatrix}0&1\\1&0\end{pmatrix};$$

(iv) *σ fixes N pointwise, and preserves μ_m, i.e.,*

$$\mu_m^{\sigma}(p_m)=\psi_m(p_m^{\sigma})=\psi_m(p_m),\ \forall p_m\in\overline{P_m^*};$$

(v)
$$\sigma\left(\begin{bmatrix}0&a\\1&0\end{bmatrix},\zeta\right)=\left(\begin{bmatrix}0&a\\1&0\end{bmatrix},(-1,-a)\zeta\right).$$

PROOF. If $g=\begin{bmatrix}a&b\\c&d\end{bmatrix}\in G$, let

$$s'(g)=\begin{cases}(\det(g),-c) & \text{if } c\neq 0,\\ (-1,d\det(g)) & \text{if } c=0,\end{cases}$$

and set

$$s(g)=\begin{cases}(c,d\det(g)) & \text{if } cd\neq 0, \text{ and the } v\text{-adic order of } c \text{ is odd},\\ 1 & \text{otherwise.}\end{cases}$$

Let $\theta\colon\bar{G}\to\bar{G}$ denote the map defined by

$$(g,\zeta)^\theta = \left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} g' \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, s(g)s(g^*)s'(g)\zeta\right)$$

where

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}^* = g^* = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Then according to theorem 3 of [7], θ is an anti-automorphism of $\bar{G}$. Moreover,

$$((g,\zeta)^\theta)^\theta = (g, \zeta(-1, \det(g))).$$

Now let $(g,\zeta)^\epsilon$ denote conjugation by $([\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}], 1)$, and set

$$(g,\zeta)^\sigma = ((g,\zeta)^\theta)^\epsilon.$$

Then it is easy to check that

$$((g,\zeta)^\sigma)^\epsilon = ((g,\zeta)^\epsilon)^\sigma (I_2, (-1, \det g)).$$

Indeed this is obvious for g in $SL_2(F)$, and for $g = [\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}]$ the computations are straightforward. Thus σ is an involutory anti-automorphism; the remaining properties of σ can be checked directly from the definitions.

Lemma 3.3. *Suppose σ is as in Lemma 3.3, and φ is in $\mathcal{H}(\psi, \mu, m)$. Then*

$$\varphi^\sigma(g) = \varphi(g).$$

Proof. Write

$$\bar{G} = \bigcup_\alpha \overline{P_m^*} g_\alpha \overline{P_m^*},$$

where α runs through a set of representatives for the $\overline{P_m^*}$-double cosets of $\bar{G}$. Because of property (iv) of Lemma 3.2, the proof of Lemma 3.3 reduces to the following:

Lemma 3.4. *Suppose σ, φ and m are as above, and g_α is fixed. Then either*

(A) $\varphi(\overline{P_m^*} g_\alpha \overline{P_m^*}) \equiv 0$, *or*

(B) *there exists g'_α in $\overline{P_m^*} g_\alpha \overline{P_m^*}$ such that $(g'_\alpha)^\sigma = g'_\alpha$.*

Proof. Let B denote the group of upper triangular matrices in G, A the group of diagonal matrices, and

$$w = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

From the Bruhat decomposition for G it follows that

$$\bar{G} = \bar{B} \cup N\bar{A}wN.$$

Note that $\overline{P_m^*} \subset N\bar{A}$. Thus the $\overline{P_m^*}$ double coset representatives can be taken to be of the form

(i) $\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$, or

(ii) $\begin{bmatrix} 1 & n_1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & a \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & n_2 \\ 0 & 1 \end{bmatrix}$.

Here a is reduced modulo U_m, and n, n_1, n_2 are reduced modulo O_F.

First consider g_α of type (i). We claim that if a does *not* belong to U_m, then $\varphi(g_\alpha) = 0$, i.e., g_α is of type (A). Indeed let $t = 1 - a$. By assumption, $t \neq \mathcal{P}^m$. But clearly there exists $b \in O_F$ such that $bt \in O_F - \mathcal{P}^m$. So on the one hand,

$$\varphi\left(\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} g_\alpha \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}^{-1}\right) = \varphi(g_\alpha),$$

and on the other,

$$\varphi\left(\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} g_\alpha \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}^{-1}\right) = \varphi\left(g_\alpha \begin{bmatrix} 1 & b(1-a) \\ 0 & 1 \end{bmatrix}\right)$$

$$= \psi_m((1-a)b)\varphi(g_\alpha).$$

Therefore, since ψ_m is non-trivial outside $\mathcal{P}^m$, $\psi_m((1-a)b) \neq 1$, and it follows $\varphi(g_\alpha) = 0$.

Now suppose

$$g_\alpha = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix},$$

and $n \notin O_F$. By the argument above, we may assume $a = 1$. Also, we may find β in U_m such that $(\beta - 1)n \in O_F - \mathcal{P}^m$. Therefore

$$\varphi(g_\alpha) = \varphi\left(\begin{bmatrix} \beta & 0 \\ 0 & 1 \end{bmatrix} g_\alpha\right) = \varphi\left(g_\alpha \begin{bmatrix} \beta & (\beta-1)n \\ 0 & 1 \end{bmatrix}\right) = \psi_m((\beta-1)n)\varphi(g_\alpha).$$

So since $(\beta - 1)n \notin \mathcal{P}^m$, we are again led to Case (A).

On the other hand, if $a \in U_m$ *and* $n \in O_F$, Lemma 3.2 imples g_α is of type (B). Therefore we need now treat only those g_α of type (ii), i.e., we suppose

$$g_\alpha = \begin{bmatrix} 1 & n_1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & a \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & n_2 \\ 0 & 1 \end{bmatrix}.$$

Note that if a is *not* a square, then Case (A) immediately applies. Indeed if a is not a square in $F^\times$,

$$\left(\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix}, 1\right) g_\alpha \left(\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix}, 1\right)^{-1} = (g_\alpha, -1)$$

for some $z \in F^\times$; cf. corollary 2.13 of [1]. Therefore, since μ (and hence φ) is assumed to be genuine,

$$\varphi(g_\alpha) = -\varphi(g_\alpha).$$

Henceforth we assume $a \in (F^\times)^2$.

If $n_1 \equiv n_2$ modulo O_F, Lemma 3.2 implies that Case (B) applies. Moreover, if there exists $\alpha \in U_m$ such that $\alpha n_1 \equiv n_2(O_F)$, then Case (B) still applies. Indeed, multiplying g_α by the matrix $\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$, with $x \in O_F$, we may suppose $\alpha n_1 = n_2$. Then multiplying g_α (on the left) by $\left(\begin{smallmatrix} \alpha & 0 \\ 0 & 1 \end{smallmatrix}\right)$ gives

$$\begin{pmatrix} 1 & n_2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \alpha y \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & n_2 \\ 0 & 1 \end{pmatrix},$$

which is obviously fixed by σ. Thus to complete the proof of Lemma 3.4, we need:

LEMMA 3.4.1. *If n_1 and n_2 are such that $\alpha n_1 \equiv n_2(O_F)$ has no solution in U_m, then Case* (A) *applies, i.e.,* $\varphi(g_\alpha) = 0$.

PROOF. Suppose there exists $t \in \mathcal{P}^m$ such that

$$tn_1, tn_2 \in O_F,$$

but

$$t(n_1 - n_2) \notin \mathcal{P}^m.$$

Then if $\beta = 1 + t$,

$$\begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix} g_\alpha \begin{pmatrix} 1 & 0 \\ 0 & \beta^{-1} \end{pmatrix} = \begin{pmatrix} 1 & (\beta-1)n_1 \\ 0 & 1 \end{pmatrix} g_\alpha \begin{pmatrix} 1 & (\beta^{-1}-1)n_2 \\ 0 & 1 \end{pmatrix},$$

with $\beta \in U_m$. Therefore

$$\varphi(g_\alpha) = \psi_m((\beta-1)n_1 + (\beta^{-1}-1)n_2)\varphi(g_\alpha),$$

and it suffices to show $(\beta - 1)n_1 + (\beta^{-1} - 1)n_2 \notin \mathcal{P}^m$. But

$$\begin{aligned}(\beta-1)n_1 + (\beta^{-1}-1)n_2 &= tn_1 - \beta^{-1}tn_2 \\ &= t(n_1 - n_2) - (\beta^{-1}-1)tn_2,\end{aligned}$$

so since $(\beta^{-1} - 1)tn_2$ belongs to $\mathcal{P}^m$, and $t(n_1 - n_2)$ does not, this is obvious.

It remains to prove:

LEMMA 3.4.2. *Suppose n_1 and n_2 are such that $\alpha n_1 \equiv n_2(O_F)$ has no solution in U_m. Then there always exists some $t \in \mathcal{P}^m$ such that*

$$tn_1, tn_2 \in O_F$$

but

$$t(n_1 - n_2) \notin \mathcal{P}^m.$$

PROOF. We proceed case by case.

Case (i). Assume $v(n_1) \neq v(n_2)$, say $v(n_1) < v(n_2)$. Of course $v(n_1) < 0$, since the hypothesis of our lemma implies neither n_1 nor n_2 is an integer. Thus there exists t_0 in $\mathcal{P}^m$ such that $t_0 n_1 \in \mathcal{P}^{m-1} - \mathcal{P}^m$, which implies $t_0 n_2 \in \mathcal{P}^m$ and $t_0(n_1 - n_2) \notin \mathcal{P}^m$.

Case (ii). Assume $v(n_1) = v(n_2) \geqq -m$. Again $v(n_1) = v(n_2) < 0$, since neither n_1 nor n_2 is in O_F. Moreover, $tn_1, tn_2 \in O_F$ for all t in $\mathcal{P}^m$ (since $n_i \in \mathcal{P}^{-m}$). If for all $t \in \mathcal{P}^m$ we also have $t(n_1 - n_2) \in \mathcal{P}^m$, then $n_1 - n_2 \in \mathcal{O}$, a contradiction. Thus $t_0(n_1 - n_2) \notin \mathcal{P}^m$ for some $t_0 \in \mathcal{P}^m$.

Case (iii). Assume $v(n_1) = v(n_2) < -m$, i.e. $n_1, n_2 \notin \mathcal{P}^{-m}$. As always, $n_1, n_2 \notin O_F$, and we can find $t_0 \in \mathcal{P}^m$ such that $t_0 n_1 \in O_F^\times$. Since $v(t_0 n_2) = v(t_0) + v(n_2) = v(t_0) + v(n_1) = 0$, we also have $t_0 n_2 \in O_F^\times$. But

$$t_0(n_1 - n_2) = t_0 n_1 \left(1 - \frac{n_2}{n_1}\right), \qquad \text{with} \quad \frac{n_2}{n_1} \quad \text{and} \quad t_0 n_1 \in O_F^\times.$$

So if $t_0(n_1 - n_2) \in \mathcal{P}^m$, we get $n_2/n_1 \in U_m$, i.e.

$$n_2 = \left(\frac{n_2}{n_1}\right) n_1, \qquad \text{with} \quad \frac{n_2}{n_1} \in U_m.$$

This contradicts our hypothesis, and completes our proof of Lemma 3.4.

PROOF OF THEOREM 3.1. Multiplication in $\mathcal{H}(\psi, \mu, m)$ is defined by the formula

$$\varphi_1 * \varphi_2(y) = \int_{Z^2\backslash \bar{G}} \varphi_1(yx^{-1}) \varphi_2(x) dx,$$

and σ is an anti-automorphism of order 2. So a well-known argument shows that

$$(\varphi_1 * \varphi_2)^\sigma = \varphi_2^\sigma * \varphi_1^\sigma.$$

But by Lemma 3.3, $\varphi_i^\sigma = \varphi_i$. Therefore $\varphi_1 \circ \varphi_2 = \varphi_2 * \varphi_1$, as desired.

REMARK 3.5. Let $\overline{P_m^{**}}$ denote the subgroup of $\bar{G}$ obtained by conjugating $\overline{P_m^*}$ by the element

$$\left(\begin{bmatrix} \tau^{-m} & 0 \\ 0 & 1 \end{bmatrix}, 1\right).$$

Then $\overline{P_m^{**}} = \overline{ZP_m^{**}}$, where

$$P_m^{**} = \left\{\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} : a, d \in U_m, b \in \mathcal{P}^{-m}\right\}.$$

Let μ_m^* denote the character of $\overline{P_m^{**}}$ obtained by composing the character μ_m of $\overline{P_m^*}$ with the natural map of $\overline{P_m^{**}}$ onto $\overline{P_m^*}$. Then Theorem 3.1 remains valid with $(\overline{P_m^{**}}, \mu_m^*)$ in place of $(\overline{P_m^*}, \mu_m)$. Moreover, for $b \in \mathcal{P}^{-m}$,

$$\mu_m^*\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \psi(b).$$

COROLLARY 3.6. *Suppose (π, V) is an irreducible admissible representation of $\bar{G}$. Given ψ, μ, and m as before, let μ_m^* denote the character of $\overline{P_m^{**}}$ defined by*

$$\mu_m^*\left(\bar{z}\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right) = \mu(\bar{z})\psi(b).$$

Then there is at most one v in V which is a μ_m^*-eigenvector for $\overline{P_m^{**}}$. More precisely, the subspace of v in V such that

$$\pi(\bar{p})v = \mu_m^*(p)v, p \in \overline{P_m^{**}}$$

is at most one-dimensional.

PROOF. The arguments are well-known. Indeed let $\mathcal{H}(\psi, \mu^*, m)$ denote the algebra of all locally constant functions on $\bar{G}$, compactly supported modulo Z^2, such that

$$\varphi(pgp') = \mu_m^*(pp')\varphi(g)$$

for all p_1p' in $\overline{P_m^{**}}$ and g in $\bar{G}$. By Theorem 3.1 (and Remark 3.5) we know that $\mathcal{H}(\psi, \mu^*, m)$ is commutative. On the other hand, π lifts to a representation of $\mathcal{H}(\psi, \mu^*, m)$ in the space of μ_m^*-eigenvectors, and the irreducibility and admissibility of π implies that this space is a finite-dimensional irreducible $\pi(\mathcal{H})$-module. So by the commutativity of $\mathcal{H}$, the Corollary results.

CONCLUDING REMARK (FOR GL_2). If μ is trivial on $\mathcal{Z}_2$, then $\mathcal{H}(\psi, \mu^*, m)$ is isomorphic to the algebra of locally constant functions on GL_2, compactly supported modulo Z, such that

$$\varphi(pgp') = \mu_m(pp')\varphi(g)$$

for all p, p' *in* P_m^{**}. In this case, the commutativity of $\mathcal{H}(\psi, \mu^*, m)$ is proved by removing "bars" everywhere in the proof just given, and using

$$g \to \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} g' \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

in place of σ. Moreover, in dealing with g_α of type (ii), we don't need to separately treat the case when a is not a square (since

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & a \\ 1 & 0 \end{bmatrix}' \begin{bmatrix} 0 & a \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a \\ 1 & 0 \end{bmatrix}$$

for *all* a).

4. Proof of uniqueness

Recall ψ is a non-trivial character ψ of F, (π, V) is any irreducible admissible representation of $\bar{G}$, and $\mu \in \Omega(\omega_\pi)$, i.e., μ is a character of $\bar{Z}$ such that

$$\pi(z)v = \mu(z)v$$

for all $z \in \bar{Z}^2$ and $v \in V$.

Our purpose in this section is to prove the following:

THEOREM 4.1. *Up to a scalar factor, there exists at most one linear functional* ℓ *on* V *such that*

$$\ell(\pi(\bar{z}n)v) = \mu(\bar{z})\psi(n)\ell(v) \tag{4.1.1}$$

for all $\bar{z} \in \bar{Z}$ *and* $n \in N$.

REMARK. We assume, without loss of generality, that ψ is trivial on O_F but not on $\mathcal{P}^{-1}$. Indeed if ψ' is an arbitrary non-trivial character, and ψ is as just described, then for some $a \in F^\times$,

$$\psi'(x) = \psi(a^{-1}x).$$

In particular, if ℓ' is a (ψ', μ) Whittaker functional for (π, V),

$$\ell(v) = \ell'\left(\pi\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}v\right)$$

defines a (ψ, μ')-functional (with $\mu'(z,\zeta) = (z,a)\mu(z,\zeta)$). Thus it suffices to prove Theorem 4.1 with ψ normalized as in Section 3.

PROOF OF THEOREM 4.1. Given an irreducible admissible representation (σ, X) of $\bar{G}$, let $(\hat{\sigma}, \hat{X})$ denote its *full* dual representation. According to Frobenius reciprocity, (σ, X) has at most one Whittaker functional iff the space of $\hat{v} \in \hat{X}$ such that

$$\hat{\sigma}(n)\hat{v} = \psi^*(n)\hat{v}, \qquad \forall n \in N^* \tag{4.1.2}$$

is at most one dimensional. Thus we want to apply Corollary 3.6 to the *contragredient* representation $\tilde{\sigma}$ — the smooth subrepresentation of $(\hat{\sigma}, \hat{X})$. More precisely, we want to use Corollary 3.6 (with $\tilde{\sigma}$ in place of π) to show there cannot be more than one linearly independent vector $\hat{v}$ in $\hat{X}$ satisfying (4.1.2).

Suppose $X(\psi^*)$ is a two-dimensional space of vectors in $\hat{X}$ satisfying (4.1.2). To show that such a space cannot exist, consider the open compact group

$$K_m = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(O_F) \colon a, d \in U_m, b \in O_F, c \in \mathcal{P}^{2m} \right\}.$$

Our group $\bar{G}$ splits over K_m, and the operator

$$I_m \colon \hat{v} \to \hat{v}_m = \int_{\bar{K}_m} \hat{\sigma}(k)\hat{v}dk \qquad \text{for} \quad m \quad \text{large}$$

is such that $\dim(I_m(X(\psi^*))) = \dim(X(\psi^*))$ for m sufficiently large. Therefore, to produce a contradiction, it suffices to prove that

$$\dim(I_m(X(\psi^*)) \leqq 1$$

for all large m.

So take m large and $\hat{v}_m$ in $I_m(X(\psi^*))$. Then $\hat{v}_m$ is smooth, and we need only show that

$$\hat{\sigma}(p)\hat{v}_m = \mu_m^*(p)\hat{v}_m, \qquad p \in \overline{P_m^{**}}.$$

But for $p \in \bar{Z}$, this is obvious from the definitions of μ_m^* and ψ^*. (Recall $\hat{\sigma}(\bar{z}n)\hat{v} = \mu(\bar{z})\psi(n)\hat{v}$ for all $z \in \bar{Z}$.) So suppose

$$p = \left(\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}, \zeta \right), \qquad \text{with } a, d \in U_m.$$

Then since such a p belongs to $\bar{K}_m$,

$$\hat{\sigma}(p)\hat{v}_m = \int_{\bar{K}_m} \hat{\sigma}(p)\hat{\sigma}(k)\hat{v}dk = \hat{v}_m = \mu_m^*(p)\hat{v}_m.$$

Suppose finally that

$$p = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}, \qquad \text{with } b \in \mathcal{P}^{-m}.$$

Then for $k = [\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}]$ in K_m,

$$pkp^{-1} = \begin{bmatrix} \alpha + b\gamma & \beta + b(\delta - \alpha) - b^2\gamma \\ \gamma & \delta - b\gamma \end{bmatrix} \in K_m.$$

Therefore

$$\hat{\sigma}(p)\hat{v}_m = \int_{\bar{K}_m} (\hat{\sigma}(p)\hat{\sigma}(k)\sigma(p^{-1}))\hat{\sigma}(p)\hat{v}dk$$

$$= \int_{\bar{K}_m} \hat{\sigma}(k')\hat{\sigma}(p)\hat{v}dk$$

$$= \psi(b)\hat{v}_m = \mu_m^*(p)\hat{v}_m,$$

and we are done.

5. Proof of existence

Given an infinite-dimensional irreducible admissible representation (π, V) of $\bar{G}$, and a fixed non-trivial character ψ of F, let $\Omega(\pi, \psi)$ denote the set of characters μ of $\bar{Z}$ such that:

(i) the restriction of μ to $\overline{Z^2}$ coincides with the "central character" of π;

(ii) there exists at least one non-zero linear functional ℓ on V such that

$$\ell(\pi(\bar{z}n)v) = \mu(\bar{z})\psi(n)\ell(v), \forall v \in V.$$

In this Section we want to prove:

THEOREM 5.1. *$\Omega(\pi, \psi)$ is non-empty.*

REMARK 5.2. *Every genuine admissible representation of $\bar{G}$ is infinite-dimensional.* Indeed suppose such a (π, V) is finite-dimensional. Since π is admissible, its kernel K is then an open normal subgroup of $\bar{G}$. In particular, K contains the subgroups N, wNw^{-1}, and $\overline{SL_2(F)}$. Thus K contains $(I_2, -1)$, contradicting the fact that π is genuine.

REMARK 5.3. Let $\mathcal{L}$ denote the complex space of functionals ℓ on V satisfying the equation

$$\ell(\pi(n)v) = \psi(n)\ell(v), \qquad n \in N. \tag{5.3.1}$$

Since $\overline{Z^2}$ is the center of $\bar{G}$, this is equivalent to the equation

$$\ell(\pi(\bar{z}n)v) = \psi(n)\omega_\pi(\bar{z})\ell(v), \qquad n \in N, \quad z \in \bar{Z}^2.$$

Now fix ω_π^* to be any extension of ω_π^{-1} to $\bar{Z}$ and define an action of $\bar{Z}$ on $\mathscr{L}$ by the formula

$$\ell^z(v) = \omega_\pi^*(z)\ell(\pi(z)).$$

Since $\bar{Z}^2$ has finite index in the abelian group $\bar{Z}$, it follows that $\Omega(\pi, \psi)$ is non-trivial if and only if $\mathscr{L}$ is. Thus it remains to prove there exists a non-trivial linear functional ℓ on V satisfying (5.3.1).

LEMMA 5.4.1. *Suppose $f(x)$ is a locally constant function on F such that for each non-trivial character ψ of F there exists an integer n such that*

$$\int_{\mathscr{P}^{-m}} f(x)\psi(-x)dx = 0 \qquad \text{for } m \geqq n.$$

Then $f(x)$ is constant.

PROOF. See the first step of the proof of lemma 3 on page 1.5 of [5].

For each non-trivial ψ on F consider the space

$$V_\psi = \left\{v \in V\colon \int_{\mathscr{P}^{-m}} \left(\pi\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}v\right)\psi(-x)dx = 0 \qquad \text{for all large } m\right\}.$$

LEMMA 5.4.2. *Suppose $V_{\psi_0} = V$ for some fixed non-trivial ψ_0. Then $V_\psi = V$ for all non-trivial ψ.*

PROOF. If $\psi(x) = \psi_0(a^{-1}x)$ for all $x \in F$, then

$$\pi\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}v \in V_\psi$$

as soon as $v \in V_{\psi_0}$. Indeed for all large m,

$$\int_{\mathscr{P}^{-m}} \psi(-x)\pi\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\pi\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}vdx = \int_{\mathscr{P}^{-m}} \psi_0(-a^{-1}x)\pi\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}\pi\begin{pmatrix}1 & a^{-1}x\\ 0 & 1\end{pmatrix}vdx$$

$$= |a|\pi\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}\int_{\mathscr{P}^{-m}} \psi_0(-x)\pi\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}vdx$$

$$= 0.$$

Therefore

$$V_\psi \supset \pi\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} V_{\psi_0} = \pi\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} V = V.$$

LEMMA 5.4.3. *For each non-trivial ψ on F, V_ψ is a proper subspace of V.*

PROOF. Suppose $V_\psi = V$ for some such ψ. Then by the last Lemma, $V_\psi = V$ for all ψ, and this implies

$$\pi\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} v = v, \qquad x \in F.$$

Indeed if $v \in V$ and ℓ is *any* linear functional on V, put

$$f_v(x) = \ell\left(\pi\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} v\right).$$

Then for all ψ,

$$\int f_v(x)\psi(-x)dx = 0$$

for m sufficiently large. So by Lemma 5.4.1, $f_v(x)$ is constant, i.e.,

$$\ell\left(\pi\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} v - v\right) = 0$$

for all $x \in F$, and since ℓ is arbitrary, our claim is established.

Now consider the stability group H of v in $\bar{G}$. By the admissibility of π, H is open. To conclude our proof of the Lemma, it suffices to show that H contains $\overline{SL_2(F)}$, since this contradicts the infinite dimensionality of π. Therefore V_ψ must be proper for all ψ.

To prove H contains $\overline{SL_2(F)}$, consider the Bruhat decomposition

$$\bar{G} = \bar{B} \cup \bar{B}wN.$$

Since H is an open subgroup, it cannot be contained in $\bar{B}$. On the other hand, we've just proved H contains N. Therefore H also contains an element of the form

$$\bar{h} = \left(\begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix}, \xi\right),$$

and since the group generated by N and $\bar{h}$ contains $\overline{SL_2(F)}$ our proof of Lemma 5.4.3 is complete.

PROOF OF THEOREM 5.1. For each non-trivial ψ, V_ψ is proper in V. Therefore there exists a non-trivial linear functional on V which vanishes on V_ψ. Our claim is that this functional provides a non-trivial solution to 5.3.1. Indeed if $y \in F$, and $v \in V$,

$$\int_{\mathcal{P}^{-m}} \psi(-x)\pi\begin{pmatrix}1 & x\\0 & 1\end{pmatrix}\left(\pi\begin{pmatrix}1 & y\\0 & 1\end{pmatrix}v - \psi(y)v\right)dx$$

$$= \int_{\mathcal{P}^{-m}} \psi(-x)\pi\begin{pmatrix}1 & x+y\\0 & 1\end{pmatrix}dx - \int_{\mathcal{P}^{-m}} \psi(y-x)\pi\begin{pmatrix}1 & x\\0 & 1\end{pmatrix}v dx.$$

Now take m so large that $y \in \mathcal{P}^{-m}$, and make the change of variables $x \to x - y$ in the *first* integral of the difference above. This gives zero, i.e.

$$\pi\begin{pmatrix}1 & y\\0 & 1\end{pmatrix}v - \psi(y)v \in V_\psi$$

for all $y \in F$. So taking ℓ as above gives

$$\ell\left(\pi\begin{pmatrix}1 & y\\0 & 1\end{pmatrix}v\right) = \psi(y)\ell(v), \qquad v \in V.$$

6. Whittaker models

Given (π, ψ), and $\mu \in \Omega(\pi, \psi)$, we've just proved there is a unique non-trivial Whittaker functional on V_π of type (ψ, μ). Denote this functional by $\mathcal{L}_\mu$, and for each v in V, define a function on $\bar{G}$ by

$$W_V^\mu(g) = \mathcal{L}_\mu(\pi(g)v).$$

LEMMA 6.1. *The Whittaker mapping*

$$v \to W_V^\mu$$

is injective, i.e.,

$$W_{V_0}^\mu(g) \equiv 0 \quad \textit{implies} \quad v_0 = 0.$$

PROOF. Let V_μ denote the subspace of v in V such that $W_v^\mu(g) \equiv 0$. Since

$$W_{\pi(g_0)v}^\mu(g) = W_v^\mu(gg_0),$$

the subspace V_μ is invariant for π. Therefore, since π is irreducible, V_μ is $\{0\}$ or V. But $\mathcal{L}_\mu$ non-trivial implies $\mathcal{L}_\mu(v^*) \neq 0$ for some v^* in V, and

$$\mathcal{L}_\mu(v^*) = W_{v^*}^\mu(1).$$

Therefore $V_\mu \neq V$.

THEOREM 6.2. *Let π be an irreducible admissible genuine representation of $\bar{G}$, ψ a non-trivial character of F, and $\mu \in \Omega(\pi, \psi)$. Then in the space of locally constant solutions of*

$$W\left(\bar{z}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) = \mu(\bar{z})\psi(x)W(g), \tag{6.2.1}$$

*there is one and only one right invariant subspace in which right translations by $\bar{G}$ define a representation of $\bar{G}$ isomorphic to π; this is the μ-*Whittaker space *of π, denoted $\mathcal{W}(\pi, \mu)$.*

PROOF. Lemma 6.1 implies that one such subspace exists, namely the image of the map $v \to W_v^\mu$. On the other hand, if V_1 is contained in the space of solutions of (6.2.1), and $\bar{G}$ operating on V_1 through right translations realizes π, then

$$W \to W(1)$$

defines a (non-trivial) (μ, ψ)-Whittaker functional which by Theorem 4.1 is unique.

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